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Euclidean, Spherical, and Hyperbolic Geometry Comparison Paper

Euclid developed five postulates as the basis from which, allegedly, all geometry could be derived. However, there was some controversy about his fifth postulate, the Parallel Postulate. This postulate states that “If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles”. Eventually, people began to discover geometries in which this postulate was false. These geometries are spherical geometry and hyperbolic geometry.

Euclid’s first four postulates are as follows: 1. There is a straight line from any point to any point. 2. A finite straight line can be produced in any straight line. 3. There is a circle with any center and any radius. 4. All right angles are equal to another. In spherical geometry, as long as “straight line” is replaced with “geodesic”, these first four postulates are true. In hyperbolic geometry, the first four postulates are also true. However, this is not true for the fifth postulate. Stated in other worlds, the fifth postulate says that for a line and a point not on that line, there is one and only one additional line that passes through the point that is parallel to the original line. In spherical geometry, all lines are geodesics, or great circles. All geodesics intersect each other. Therefore, on a sphere, there are no parallel lines. When it comes to the fifth postulate, hyperbolic geometry is the complete opposite of spherical geometry. In hyperbolic geometry, there are infinitely many lines that pass through a given point that are parallel to the original line. In this case, parallel means that the lines never intersect, not that they are an equal distance apart.

In each geometry, while some characteristics for triangles stay the same, there are major characteristics that differ. In each geometry, a triangle has three sides. This is the one characteristic that is true for all three geometries. In Euclidean geometry, there can be similar triangles that aren’t congruent. However, in both spherical geometry and hyperbolic geometry, there are no similar triangles that aren’t congruent. Also in these two geometries, the Pythagorean theorem, which states the sum of the squares of the lengths of the two legs of a right triangle is equal to the square of the length of the hypotenuse of that triangle, is false. One major difference between triangles in all three geometries is the sum of the angles of a triangle. In Euclidean geometry, the sum of the angles of a triangle is exactly 180 degrees. In spherical geometry, the sum of the angles of a triangle is greater than 180 degrees but less than 900 degrees. In hyperbolic geometry, the sum of the angles of a triangle is less than 180 degrees. In Euclidean geometry, in order to find the area of a triangle, one has to multiply the base of the triangle by the height of the triangle and divide it by two. For spherical geometry, in order to find the area for a triangle, one has to use Girard’s theorem. In Girard’s theorem, 180 subtracted from the sum of the angles gives the excess number of degrees of the triangle. This excess divided by 720 and multiplied by the area of the sphere is equal to the area of the triangle. In hyperbolic geometry, there is a fixed number so that the area of a triangle is less than that number. Also in hyperbolic geometry, there are no triangles that you can draw a circle around.

In each geometry, there are also similarities and differences between other polygons. In all three geometries, a polygon is a closed figure. In both Euclidean and hyperbolic geometry, a polygon must have a minimum of three sides. However, in spherical geometry, a polygon has a minimum of two sides. This two-sided shape is called a lune which is formed when two lines intersect twice. In Euclidean and hyperbolic geometry, no two lines intersect twice, so there are no two-sided polygons. In hyperbolic geometry, rectangles do not exist.

In each geometry, definitions are very important. This is because in each geometry, each shape and math term has its own definition because of the conditions and rules that change with each geometry. This is shown with the first four of Euclid’s postulates. If “straight line” is not replaced with “geodesic”, the postulates are not true for spherical geometry. Another example where definitions are important is with a square. In Euclidean geometry, a square is a shape with equal side lengths and four 90 degree angles. However, this is not the definition of a square in spherical or hyperbolic geometry. In both spherical and hyperbolic geometry, a square is a shape with four equal angles, not four 90 degree angles. In Euclidean geometry, if a square is cut in half, it creates two triangles, each with an angle sum of 180 degrees. However, in spherical geometry, the sum of the angles of a triangle is greater than 180 degrees and in hyperbolic geometry, the sum of the angles of a triangle is less than 180 degrees which means that a square cannot be a shape with four 90 degree angles.

There are many places where applications of each geometry can be found. In Euclidean geometry, applications include buildings, distances between locations, and daily use of geometry. In spherical geometry, one application includes airplane routes. Airplanes take great circle routes to get to a destination rather than traveling in a straight path, because a geodesic is the shortest distance between two points. In hyperbolic geometry, it is believed that the shape of the universe may be hyperbolic. Einstein’s theory of relativity also used a hyperbolic model.

One place where each geometry has different requirements is with tiling. If one looks at their kitchen, basement, or bathroom, they may see square tiling. However, how would one use square tiles on a sphere or hyperbolic shape? One could tile a spherical shape using 120 degree triangles, 90 degree triangles, 72 degree triangles, 120 degree quadrilaterals, and 120 degree pentagons. One may see these shapes being used for tiling in objects like soccer balls. In hyperbolic geometry, objects used for tiling include triangles, hexagons, and pentagons. The farther away the tiling shape is from the center of the hyperbolic shape, the smaller it appears. Although the world in which we live primarily exhibits Euclidean geometry, both spherical and hyperbolic geometries are very important and provide a different perspective on the world as we know it!